Analytical solutions to one-dimensional dissipative and discrete chaotic dynamics

D. Pingel and P. Schmelcher
Theoretische Chemie, Physikalisch-Chemisches Institut, Universität Heidelberg, Im Neuenheimer Feld 253, D-69120 Heidelberg, Germany

F. K. Diakonos
Department of Physics, University of Athens, GR-15771 Athens, Greece

(Received 12 December 1997)

Analytical solutions to the chaotic and ergodic motion of a certain class of one-dimensional dissipative and discrete dynamical systems are derived. This allows us to obtain exact expressions for physical properties such as the time correlation function. We illustrate our solutions by means of a few examples for which conventional numerical trajectory calculations fail to predict the correct behavior. [S1063-651X(98)01207-0]

PACS number(s): 05.45.+b, 95.10.Fh

I. INTRODUCTION

Chaotic behavior is a characteristic feature of the overwhelming majority of deterministic nonlinear dynamical systems [1]. Classical chaos manifests itself in the exponential sensitivity of the trajectories with respect to the initial conditions and has nicely been understood as the very complex topological process of everlasting stretching and folding of the motion. In view of the complexity of chaotic trajectories it appears hard to imagine that chaotic motion could be described by means of closed analytical formulas. To our knowledge there exists only one exception, i.e., the dynamical law, which allows the representation of its chaotic trajectories for arbitrarily long times in a closed analytical form: The solutions of the logistic map in the ergodic chaotic limit are given by the celebrated Pincherlé relation [2]. In contrast to our lack of knowledge concerning exact expressions for chaotic motion there exists a well-founded and justified interest in obtaining solutions to ergodic and chaotic behavior.

On the one hand it is clear that the frequently used numerical solutions of chaotic dynamical systems do not yield one and the same trajectory for long time scales [3]. Nevertheless due to the shadowing argument statistical quantities of chaotic ergodic systems with and without external noise can in many cases be approximately obtained through a numerical investigation [4,5]. However, there are exceptions, such as the case when a Lyapunov exponent fluctuates about zero and this is expected to be common in simulations of higher-dimensional systems [6]. It is therefore highly desirable to make exact properties accessible: If the exact trajectory could be derived this would be an excellent starting point for calculating the exact correlation function, invariant measure, Lyapunov exponent, or other quantities [7]. In particular it would also offer the possibility of determining the exact long-time behavior of relevant physical quantities. Even more important may be the fact that analytical representations of chaotic motion are of principal interest and can certainly enhance our understanding of the complexity of chaotic dynamics in general. Typical questions that could then be addressed are as follows: How does the exponential sensitivity with respect to the initial conditions come about? Are there any characteristic (scaling) properties and/or self-similar structures of the analytical expressions that are responsible for the complex stretching and folding process of chaotic dynamics?

The current investigation represents a step in the above direction and provides closed analytical formulas for two classes of one-dimensional unimodal, dissipative, ergodic, and chaotic maps of the interval, each containing an infinite number of members. In particular we will derive exact solutions for two classes of maps: the conjugates to both the symmetric as well as asymmetric tent maps. Using these solutions we will derive exact expressions for important physical quantitites such as the correlation function. Some specific examples are discussed in detail, thereby demonstrating the deviation of exact and numerically determined properties.

II. SOLUTIONS TO THE CHAOTIC DYNAMICS OF THE GENERAL CONJUGATES OF THE SYMMETRIC TENT MAP

A. Trajectories

Let us begin our investigation by considering the symmetric tent map (STM) [3]. The nth iterate of the STM $t^{(n)}(x)$ possesses 2^n monotonicity intervals with alternating constant slopes 2^n and -2^n. The zeros and maxima are at the positions $k/2^n$ with $k=1,\ldots,2^n$. This makes it possible to represent $t^{(n)}(x)$ in the form

$$t^{(n)}(x)=\begin{cases} 2^n x-2(k-1); & x\in\left[\dfrac{2(k-1)}{2^n},\dfrac{2k-1}{2^n}\right] \\ -2^n x+2k; & x\in\left[\dfrac{2k-1}{2^n},\dfrac{2k}{2^n}\right]. \end{cases} \quad (1)$$

The above formula can be recast into the following very simple expression

$$t^{(n)}(x)=1-|(2^n x \mod 2)-1|, \quad (2)$$

where mod is the modulo operation. We consider now the family of maps $g(x)=u\circ t\circ u^{-1}(x)$, which are obtained from the STM by conjugation with an invertible and differentiable function $u(x)$, which maps the unit interval onto itself and obeys $u(0)=0$, $u(1)=1$ [8]. The nth iterates of the maps $\{g(x)\}$ are then given by

$$g^{(n)}(x)=(u\circ t\circ u^{-1})\circ(u\circ t\circ u^{-1})\circ\cdots\circ(u\circ t\circ u^{-1})(x)=u\circ t^{(n)}\circ u^{-1}(x) \quad (3)$$

and therefore $g^{(n)}(x)$ takes on the following appearance:

$$g^{(n)}(x)=u(1-|\{[2^n u^{-1}(x)] \mod 2\}-1|), \quad (4)$$

which represents a closed form analytical solution to the dynamics of the maps conjugated to the STM. The general conjugation yields a variety of symmetric as well as nonsymmetric maps. Imposing the restrictive condition $u(x)=1-u(1-x)$ for the conjugating functions $u(x)$ we obtain the special class of the so-called doubly symmetric maps for which both the invariant density as well as the map is symmetric. In particular if we use the specific conjugation $u_l(x)=\sin^2(\pi x/2)$ and correspondingly $u_l^{-1}(x)=(1/\pi)\arccos(1-2x)$ we arrive at the logistic map $l(x)=4x(1-x)$ and after a little algebra at the Pincherlé relation $l^{(n)}(x)=\frac{1}{2}\{1-\cos[2^n\arccos(1-2x)]\}$ for the nth iterate of the logistic map.

Equation (4) nicely demonstrates the exponential sensitivity with respect to the initial conditions as well as the infinite process of stretching and folding. The stretching process takes place through the multiplication with an exponential factor (2^n) and the folding through the mod function, which cuts with increasing iterations an increasing number of digits making a more detailed specification of the initial conditions necessary in order to describe the actual motion.

B. Exact properties and examples

Since the Lyapunov exponent is invariant with respect to conjugation [3,9] all members of the above-discussed class of maps possess the same Lyapunov exponent $\lambda=\ln 2$. In particular, since $\mu(x)=x$ is the invariant measure of the STM, the measure of the conjugated maps is $\mu(x)=u^{-1}(x)$ and, therefore, varies widely with changing conjugating function. Next let us derive analytical expressions for another important physical quantity, namely, the correlation function, which is defined by $C(n)=\hat{C}(n)-\overline{x}^2$ with $\hat{C}(n)=\int_0^1 g^{(n)}(x)x\,d\mu(x)$ and the mean value $\overline{x}=\int_0^1 x\,d\mu(x)$. Let us denote the positions of the zeros and maxima of $g^{(n)}(x)$ by $\{x_{2k}\}$ and $\{x_{2k-1}\}$, respectively. Using the fact that $g^{(n)}(x)$ is conjugate to some "original" map $h^{(n)}(x)$ as well as the relation $u(y_i)=x_i$ the correlation function can be decomposed as follows:

$$\hat{C}(n)=\sum_{k=1}^{2^{n-1}}\hat{C}_k(n) \quad \text{with} \quad \hat{C}_k(n)=\int_{y_{2k}}^{y_{2k+1}}u(x)u(h^{(n)}(x))dx+\int_{y_{2k+1}}^{y_{2(k+1)}}u(x)u(h^{(n)}(x))dx, \quad n>0. \quad (5)$$

In order to provide exact correlation functions for some specific classes of maps we now specialize to the case that $h(x)$ is given by the STM. Using Eq. (1) we can derive the following simple structure for the terms of the sum of the correlation function:

$$\hat{C}_k(n)=\int_{(k-1)/2^{n-1}}^{(2k-1)/2^n}u(x)u(2^n x-2(k-1))dx+\int_{(2k-1)/2^n}^{k/2^{n-1}}u(x)u(-2^n x+2k)dx. \quad (6)$$

In general the terms $\hat{C}_k(n)$ can be evaluated analytically as we shall show in the following by means of a few examples. Let us first choose the conjugation $u(x)=x^{1/(1+\beta)}$, which results in the invariant measure $\mu(x)=x^{1+\beta}$ and the corresponding normalized power law density $\rho(x)=(\beta+1)x^{\beta}$, $\beta>-1$. Due to their scaling properties power law densities are of particular interest for physical systems with critical or self-similar behavior. Using Eqs. (5) and (6) we arrive after some algebra at the following closed form analytical expressions for the corresponding correlation function:

$$C(n)=\left(\frac{\beta+1}{\beta+3}\right)\left(\frac{1}{2^n}\right)^{(\beta+2)/(\beta+1)}+2^{[2-n(\beta+2)]/(\beta+1)}+2^{(1-n)(\beta+2)/(\beta+1)}\sum_{k=2}^{2^{n-1}}\left\{\frac{1}{2}(k-1)^{1/(1+\beta)}B\left(1,\frac{\beta+2}{\beta+1}\right)\times{}_2F_1\left(\frac{-1}{1+\beta},\frac{2+\beta}{1+\beta};\frac{3+2\beta}{1+\beta};\frac{-1}{2(k-1)}\right)+2^{1/(1+\beta)}k^{(\beta+3)/(\beta+1)}B\left(\frac{2+\beta}{1+\beta},\frac{2+\beta}{1+\beta},\frac{1}{2k}\right)\right\}-\left(\frac{\beta+1}{\beta+2}\right)^2, \quad (7)$$

where $B(a,b)$ and $B(a,b,x)$ denote the complete and incomplete unnormalized Beta function, respectively, and ${}_2F_1(a,b,c,x)$ is the hypergeometric function. For the particular case $\beta=0$ the above expression reduces to the δ correlation $C(n)\propto\delta_{n0}$ as expected. Figure 1(a) shows the absolute value of the above correlation function for different values of the power β. The short time as well as long time behavior change somewhat with changing parameter β. The asymptotic behavior $(n\rightarrow\infty)$ of $C(n)$ is an exponential decay. The decay constant τ can be determined using the Euler-MacLaurin sum formula [10] for the asymptotic expansion of Eq. (7). It turns out that for $\beta>0$ $\tau=[(\beta+2)/(\beta+1)]\ln 2$ while for $-1<\beta<0$ we have $\tau=2\ln 2$ independent of β. Having obtained the exact correlation function for the

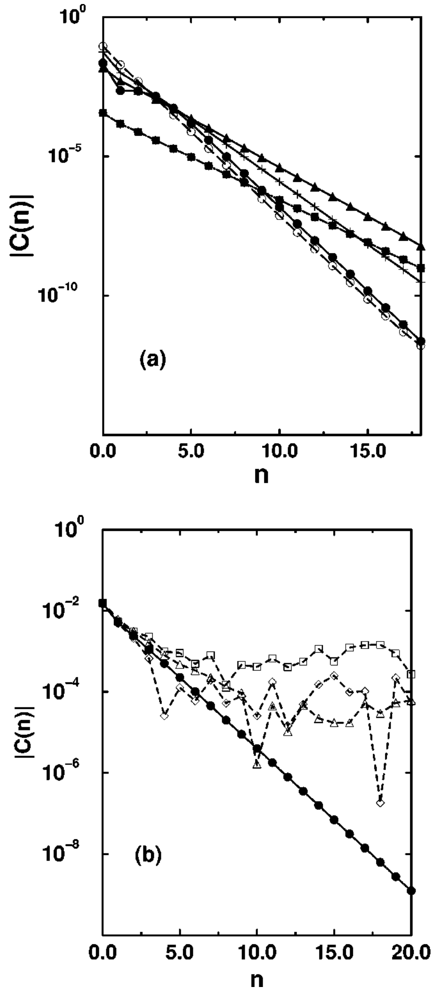

FIG. 1. (a) The logarithm of the absolute value of the correlation function for the conjugation $u(x)=x^{1/(1+\beta)}$ of the STM for different values of the parameter β. The values $\beta=-0.95,-0.5,+1.0,+5.0,+50.0$ correspond to the curves with the full circles, circles, plus, full triangles, and full boxes, respectively. (b) The exact correlation function for $\beta=5.0$ (solid line) together with the correlation functions resulting from numerical trajectory calculations (broken lines) with 10^3 (squares), 10^4 (diamonds), and 10^5 (triangles) points.

above class of maps with power law density we are in the position to compare these results with those of numerical trajectory simulations. Figure 1(b) shows the comparison of the exact correlation function for $\beta=5.0$ (full circles) with the results of numerical trajectory calculations using 10^3 (squares), 10^4 (diamonds), and 10^5 (triangles) points. Obviously there is a strong inherent deviation and in particular it can be observed that an enhancement of the number of points of the numerically calculated trajectory does not yield an improvement in the statistical accuracy of the corresponding correlation functions. This is due to the fact that a naive numerical simulation loses for the above dynamical systems very rapidly the original trajectory and due to numerical inaccuracies gets trapped on a certain ''orbit.''

Finally let us provide one more example out of a large number accessible by the above given analytical formulas. We choose as a conjugating function $u(x)=\sin[(\pi/2)x]$, which results in the measure $\mu(x)=(2/\pi)\arcsin(x)$ and the invariant density $\rho(x)=(2/\pi)1/\sqrt{1-x^2}$. A straightforward but tedious calculation yields the terms $C_k(n)$, which can be summed up to the final beautiful result of the correlation function

$$C(n)=\frac{2^{n+1}}{\pi(2^{2n}-1)}\cot\left(\frac{\pi}{2^{n+1}}\right)-\left(\frac{2}{\pi}\right)^2, \tag{8}$$

which shows with respect to its asymptotic behavior ($n\rightarrow\infty$) an exponential decay with a decay constant 2 ln2.

III. SOLUTIONS TO THE CHAOTIC DYNAMICS OF THE CONJUGATES OF THE ASYMMETRIC TENT MAP

Solving the problem of the chaotic and ergodic dynamics generated by the general class of maps conjugated to the asymmetric tent maps (ATM) [3] is a much more intricate task than the corresponding solution of the dynamics of the maps conjugate to the STM. In order to develop the necessary concepts and techniques we therefore proceed in several steps.

A. The concept of mode superposition

The knowledge of the position of the minima and/or maxima, i.e., the length of the monotonicity intervals, of the nth iterate of the ATM is central to the derivation of an analytical expression for the nth iterate of the ATM. In the present subsection we therefore provide a method that allows one to determine the length $L_n(i)$ of the ith monotonicity interval (we count the intervals starting from the origin) by a decomposition technique of the corresponding frequency distribution $r_n(i)$ (see below) into independent modes.

The ATM is unimodal with constant slopes $1/p$ and $-1/q$ $[q=(1-p)]$ on the two monotonicity intervals, respectively. First of all we remark that each iteration process $n\rightarrow n+1$ divides a monotonicity interval of the nth iterate into two monotonicity intervals of the $(n+1)$th iterate, which possesses positive and negative slopes in these intervals, respectively. The ratios of their lengths are given by $p:q$ or $q:p$ depending on whether the nth iterate of the ATM on the original monotonicity interval possesses a positive or a negative slope, respectively. In order to specify the length of the ith monotonicity interval we do not need such detailed information like the symbolic sequence [3] but only the frequency of the occurrence of the factors p and q during the iteration process. The number of different lengths for the monotonicity intervals is therefore much smaller than the number of monotonicity intervals themselves. The length of the ith monotonicity interval of the nth iterate of the ATM is given by

$$L_n(i)=p^{n-r_n(i)}q^{r_n(i)}, \tag{9}$$

where $r_n(i)$ is the frequency of the occurrence of the factor q during the branching process into the final ith monotonicity interval of the nth iterate. Using the above-described properties of the branching of the monotonicity intervals it can be shown that $r_n(i)$ obeys the following recursion formula:

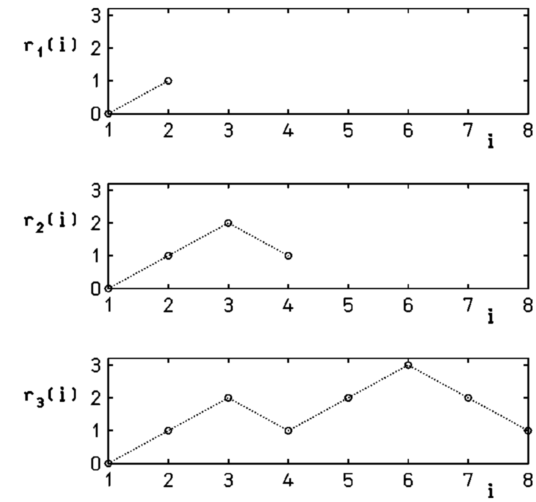

FIG. 2. The frequency distributions $r_1(i)$, $r_2(i)$, and $r_3(i)$ of the ATM.

$$r_1(1)=0,$$

$$r_{n+1}(i)=\begin{cases} r_n(i), & 1\leqslant i\leqslant 2^n \\ r_n[2^{n+1}-(i-1)]+1, & 2^n+1\leqslant i\leqslant 2^{n+1}, \end{cases} \tag{10}$$

$r_{n+1}(i)$ can therefore be obtained from $r_n(i)$ through a reflection σ_n of the function $r_n(i)$ $(i=1,\ldots,2^n)$ with respect to the vertical axis located at $2^n+\frac{1}{2}$ and subsequent addition τ_n of 1 to the attached part of $r_{n+1}(i)$, which yields in total $r_{n+1}(i)=(\tau_n\circ\sigma_n)r_n(i)$. This process is illustrated in Fig. 2 for the three functions $r_1(i),r_2(i),r_3(i)$.

In the following we derive a decomposition of the functions $r_n(i)$ into different modes, i.e., $r_n(i)$ can then be described by a superposition of these modes. Let us introduce $n-1$ modes $M_{n,m}(i),m=2\ldots n$ and the additional mode N_n. Each mode $M_{n,m}(i)$ is defined on the finite support $i=1,\ldots,2^n$ and oscillates as a function of i with the period $T_m=2^m$. In addition it possesses a phase $\phi(M_{n,m})=2^{m-2}$ which describes the shifting of the oscillations on the i axis. The mode N_n is characterized by the period 2^n and the phase $\phi(N_n)=2^{n-1}$. The equally weighted superposition of these modes yields the quantity $r_n(i)$, i.e., we have

$$r_n(i)=\sum_{m=2}^{n} M_{n,m}(i)+N_n(i). \tag{11}$$

Figure 3 illustrates the three modes $M_{3,2}(i)$, $M_{3,3}(i)$, and $N_3(i)$, which are necessary in order to build up $r_3(i)$. In particular we now briefly verify that the above decomposition obeys the recursion formula given in Eq. (10). The action of the reflection σ_n onto the modes $M_{n,m}(i)$ is given by

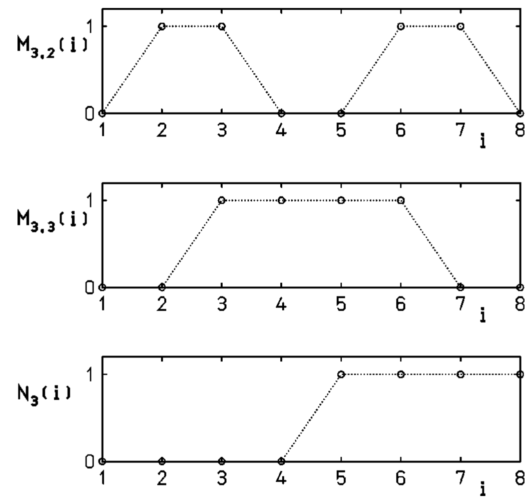

FIG. 3. The modes $M_{3,2}(i)$, $M_{3,3}(i)$, and $N_3(i)$ of the ATM.

$$M_{n,m}\overset{\sigma_n}{\longrightarrow}M_{n+1,m}, \quad N_n\overset{\sigma_n}{\longrightarrow}M_{n+1,n+1}. \tag{12}$$

The application of the subsequent addition τ_n to the reflected modes corresponds to the inclusion of the mode N_{n+1}. In total we therefore arrive at the desired relation

$$r_n(i)=\sum_{m=2}^{n} M_{n,m}(i)+N_n(i) \overset{\tau_n\circ\sigma_n}{\longrightarrow}$$
$$r_{n+1}(i)=\sum_{m=2}^{n+1} M_{n+1,m}(i)+N_{n+1}(i). \tag{13}$$

In the following subsection we provide construction principles and closed analytical formulas for the modes $M_{n,m}(i)$ and N_n.

B. Construction of the modes

Each mode function is defined on 2^n natural numbers and their continuation and representation on the real axis is therefore not unique. In the present subsection we provide two different representations for the mode functions $M_{n,m}(i)$ and $N_n(i)$. The first one is characterized by the application of the step function to trigonometric functions and the second one uses polynomials and their periodic continuation in order to describe the modes.

Looking at Fig. 3 suggests a representation of the modes by the action of the step function $(\Theta(x)=\{1:x\geqslant 0;0:x<0\})$ onto an oscillating function. Choosing the sin function for the oscillating part we can adapt the periodicity and phase in order to obtain the modes $M_{n,m}(i)$ and $N_n(i)$, respectively. They take on the following appearance:

$$M_{n,m}(i)=\Theta\left(\sin\left(\pi\frac{2i-(2^{m-1}+1)}{2^m}\right)\right), \quad N_n(i)=\Theta\left(\sin\left(\pi\frac{2i-(2^n+1)}{2^n}\right)\right). \tag{14}$$

For the function $r_n(i)$ we therefore arrive at the expression

$$r_n(i)=\sum_{m=2}^{n}\Theta\left(\sin\left(\pi\frac{2i-(2^{m-1}+1)}{2^m}\right)\right)+\Theta\left(\sin\left(\pi\frac{2i-(2^n+1)}{2^n}\right)\right). \tag{15}$$

Introducing the above formula for $r_n(i)$ in Eq. (9) we have an analytical expression for the length of the ith monotonicity interval. The only feature that could be seen as a disadvantage of the above representation is its discontinuity through the introduction of the step function. It is therefore desirable to gain a second representation of the mode functions that should have the property of being smooth with respect to the continuation onto the entire real axis. We therefore use polynomials whose coefficients will be adapted correspondingly. The introduction of a sin function yields then a periodic continuation of the polynomials and allows one to adapt to the functional form of the modes via a corresponding scaling operation and phase shift.

The first step in the derivation of a second, smooth, representation of the modes $M_{n,m}(i)$ and $N_n(i)$ is the construction of a polynomial with suitable properties. To achieve this, our starting point is a polynomial $p_m(x)$, which possesses at 2^{m-1} points x_i^+ the value $+1$ and at other 2^{m-1} points x_i^- the value -1,

$$p_m(x_1^+)=p_m(x_2^+)=\cdots=p_m(x_{2^{m-1}}^+)=1,\quad p_m(x_1^-)=p_m(x_2^-)=\cdots=p_m(x_{2^{m-1}}^-)=-1. \tag{16}$$

For the time being the domain of definition of the above introduced polynomial includes 2^m integer values and possesses 2^{m-1} zeros. Such a polynomial can be written as a sum of 2^m terms, each of the terms thereby acquires the value $+1$ or -1 at exactly one position given by x_i^+ or x_i^- , respectively, whereas at all other positions x_j^+ , x_j^- ,$i\neq j$, *it vanishes. These properties are guaranteed if each term consists of a normalized product of linear factors and we therefore arrive at the following expression for the polynomial* $p_m(x)$:

$$p_m(x)=\sum_{i=1}^{2^{m-1}}\frac{(x-x_i^-)}{(x_i^+-x_i^-)}\prod_{\substack{j=1\\ j\neq i}}^{2^{m-1}}\frac{(x-x_j^+)(x-x_j^-)}{(x_i^+-x_j^+)(x_i^+-x_j^-)}-\sum_{i=1}^{2^{m-1}}\frac{(x-x_i^+)}{(x_i^--x_i^+)}\prod_{\substack{j=1\\ j\neq i}}^{2^{m-1}}\frac{(x-x_j^+)(x-x_j^-)}{(x_i^--x_j^+)(x_i^--x_j^-)}. \tag{17}$$

The points of support x_i^+ , x_i^- ought to be equidistant in the unit interval $[0,1]$ and should be arranged symmetrically with respect to 0, i.e., we have

$$x_i^-=-x_i^+=x_i,\quad x_i=\frac{2i-1}{2^m}. \tag{18}$$

After a little algebra this leads to an essential simplification of the polynomials (17)

$$p_m(x)=\sum_{i=1}^{2^{m-1}}\frac{x}{x_i}\prod_{\substack{j=1\\ j\neq i}}^{2^{m-1}}\frac{x^2-x_j^2}{x_i^2-x_j^2}. \tag{19}$$

As a next step we perform a periodic continuation of the polynomial $p_m(x)$ by substituting

$$x\to\sin(\pi x),\quad x_i\to\sin(\pi x_i). \tag{20}$$

In order to describe the modes with the constructed polynomials we have to adapt the frequency as well as phase of the oscillating periodic functions defined by Eqs. (19) and (20). In addition a shift is performed in order to make the points of support equal to integers on which the modes are defined. Finally we arrive at the following expressions for the individual modes:

$$M_{n,m}(x)=\frac{1}{2}\left\{1+\sum_{i=1}^{2^{m-1}}\frac{\sin\{\pi[2x-(2^{m-1}+1)]/2^m\}}{\sin[\pi(2i-1)/2^m]}\prod_{\substack{j=1\\ j\neq i}}^{2^{m-1}}\frac{\sin^2\{\pi[2x-(2^{m-1}+1)]/2^m\}-\sin^2[\pi(2j-1)/2^m]}{\sin^2[\pi(2i-1)/2^m]-\sin^2[\pi(2j-1)/2^m]}\right\}, \tag{21}$$

$$N_n(x)=\frac{1}{2}\left\{1+\sum_{i=1}^{2^{n-1}}\frac{\sin\{\pi[2x-(2^n+1)]/2^n\}}{\sin[\pi(2i-1)/2^n]}\prod_{\substack{j=1\\ j\neq i}}^{2^{n-1}}\frac{\sin^2\{\pi[2x-(2^n+1)]/2^n\}-\sin^2[\pi(2j-1)/2^n]}{\sin^2[\pi(2i-1)/2^n]-\sin^2[\pi(2j-1)/2^n]}\right\} \tag{22}$$

and for the frequency distribution $r_n(i)$ of the ith monotonicity interval

$$r_n(i)=\sum_{m=2}^{n}\frac{1}{2}\left\{1+\sum_{i=1}^{2^{m-1}}\frac{\sin\{\pi[2x-(2^{m-1}+1)]/2^m\}}{\sin[\pi(2i-1)/2^m]}\prod_{\substack{j=1\\ j\neq i}}^{2^{m-1}}\frac{\sin^2\{\pi[2x-(2^{m-1}+1)]/2^m\}-\sin^2[\pi(2j-1)/2^m]}{\sin^2[\pi(2i-1)/2^m]-\sin^2[\pi(2j-1)/2^m]}\right\}$$
$$+\frac{1}{2}\left\{1+\sum_{i=1}^{2^{n-1}}\frac{\sin\{\pi[2x-(2^n+1)]/2^n\}}{\sin[\pi(2i-1)/2^n]}\prod_{\substack{j=1\\ j\neq i}}^{2^{n-1}}\frac{\sin^2\{\pi[2x-(2^n+1)]/2^n\}-\sin^2[\pi(2j-1)/2^n]}{\sin^2[\pi(2i-1)/2^n]-\sin^2[\pi(2j-1)/2^n]}\right\}. \tag{23}$$

This concludes our construction of the modes.

C. Analytical representation of the chaotic dynamics

The knowledge of the frequency distribution $r_n(i)$ is a key ingredient for the derivation of analytical solutions to the chaotic motion of the conjugates of the ATM. Arbitrarily high iterates are now accessible in closed analytical form via the following procedure.

First of all we give some important quantities related to the nth iterated map:

(i) The length of the ith monotonicity intervals is given by

$$L_n(i)=p^{n-r_n(i)}q^{r_n(i)}. \tag{24}$$

(ii) The slope in the ith monotonicity interval is

$$S_n(i)=p^{r_n(i)-n}q^{-r_n(i)}(-1)^{(i-1)}. \tag{25}$$

(iii) The left and right boundary $G_n^L(i)$ and $G_n^R(i)$ of the ith monotonicity interval can be obtained by summation of the lengths of the corresponding intervals:

$$G_n^L(i)=\sum_{j=1}^{i-1}L_n(j)=\sum_{j=1}^{i-1}p^{n-r_n(j)}q^{r_n(j)},$$
$$G_n^R(i)=\sum_{j=1}^{i}L_n(j)=\sum_{j=1}^{i}p^{n-r_n(j)}q^{r_n(j)}. \tag{26}$$

(iv) Definition of the index function: Let us define a function $I_n:[0,1]\to\{1,\ldots,2^n\}$ whose value $I_n(x)$ tells us the number of the monotonicity intervals in which x is contained. The index function can be described with the help of the step function $\Theta(x)$ in the following way

$$I_n(x)=\sum_{i=1}^{2^n}\Theta(x-G_n^L(i))=\sum_{i=1}^{2^n}\Theta\left(x-\sum_{j=1}^{i-1}p^{n-r_n(j)}q^{r_n(j)}\right). \tag{27}$$

We are now in the position of formulating the nth iterate $t_p^{(n)}(x)$ of the ATM in a closed analytical expression:

$$t_p^{(n)}(x)=\{x-G_n^L(I_n(x))\}S_n(I_n(x))+\tfrac{1}{2}[1+(-1)^{I_n(x)}]. \tag{28}$$

Inserting Eqs. (26) and (27) into the above expression yields

$$t_p^{(n)}(x)=\left[x-\sum_{j=1}^{I_n(x)-1}p^{n-r_n(j)}q^{r_n(j)}\right]p^{r_n[I_n(x)]-n}q^{-r_n[I_n(x)]}$$
$$\times(-1)^{I_n(x)-1}+\frac{1}{2}[1+(-1)^{I_n(x)}]. \tag{29}$$

Similar to the case of the STM we obtain the nth iterate $g_{t_p}^{(n)}(x)$ of the maps conjugated to the ATM by the following decomposition

$$g_{t_p}^{(n)}(x)=u\circ t_p\circ u^{-1}\circ u\circ t_p\cdots\circ t_p\circ u^{-1}(x)$$
$$=u\circ t_p^{(n)}\circ u^{-1}(x), \tag{30}$$

which gives together with Eq. (29) our final result:

$$g_{t_p}^{(n)}(x)=u\left\{\left[u^{-1}(x)-\sum_{j=1}^{I_n(u^{-1}(x))-1}L_n(j)\right]S_n[I_n(u^{-1}(x))]\right.$$
$$\left.+\tfrac{1}{2}[1+(-1)^{I_n(u^{-1}(x))}]\right\}. \tag{31}$$

Equation (31) can be used to calculate arbitrarily high iterates of the conjugates of the ATM. As an example we show in Fig. 4 the 8th iterate of the conjugate $u(x)=\sin^2(\pi x/2)$ of the ATM for $p=0.9$, which gives an idea of the scaling structures contained in the above analytical formula.

The invariant measure of the conjugates of the ATM is given by $\mu(x)=u^{-1}(x)$ and the Lyapunov exponent is known to be $\lambda=-p\ln(p)-(1-p)\ln(1-p)$ [11]. The above Eq. (31) can be used to calculate the correlation functions of the conjugates of the ATM as well as any other physical property.

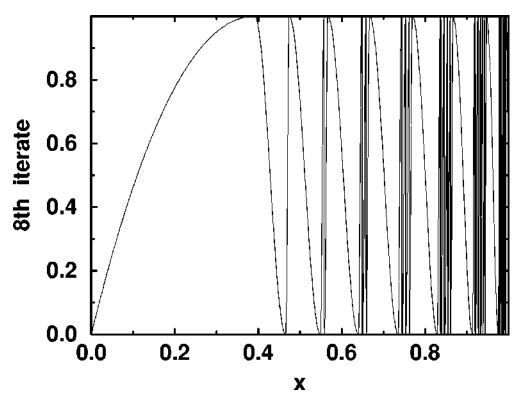

FIG. 4. The eighth iterate of the conjugate $u(x)=\sin^2(\pi x/2)$ of the ATM for $p=0.9$ calculated through Eq. (31).

IV. CONCLUSIONS

We have presented analytical solutions to the chaotic and ergodic dynamics of two classes of noninvertible single humped maps: the conjugates of the symmetric as well as asymmetric tent map. With the exception of the iterates of the logistic map this is, to our knowledge, the first time that closed analytical representations of chaotic trajectories are derived. This enables us to calculate exact physical quantities, such as the time correlation function, even in those cases for which numerical trajectory calculations fail to predict the correct long time behavior. The complex stretching and folding process of the iteration of the dynamical system is clearly revealed within our analytical approach through the combination of conjugating functions, scale transformations, and cutoff operations.

The key to the construction of the analytical solutions is the complex superposition of modes connected with the monotonicity intervals of the nth iterate of the dynamical system. We conjecture that this holds also for the general case of arbitrary single humped maps and might be a hint towards the construction of their solutions. The boundaries of the monotonicity intervals are then given by the preimages of the maximum of the dynamical law.

ACKNOWLEDGMENT

One of the authors (F.K.D.) gratefully acknowledges financial support by the European Union.

[1] H. G. Schuster, *Deterministic Chaos* (VCH, Weinheim, 1994).
[2] S. Pincherlé, Rend. della Real Acad. dei Lincei **29**, 329 (1920); S. Rabinovich, G. Berkolaiko, S. Buldyrev, A. Shehter, and S. Havlin, Int. J. Bifurcation Chaos Appl. Sci. Eng. **7**, 837 (1997).
[3] J. L. Cauley, *Chaos, Dynamics and Fractals: An Algorithmic Approach to Deterministic Chaos*, Cambridge Nonlinear Science Series Vol. 2 (Cambridge University Press, Cambridge, 1994).
[4] C. Grebogi, S. M. Hammel, J. A. Yorke, and T. Sauer, Phys. Rev. Lett. **65**, 1527 (1990).
[5] T. Sauer, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **79**, 59 (1997).
[6] S. Dawson, C. Grebogi, and T. Sauer, Phys. Rev. Lett. **73**, 1927 (1994).
[7] F. K. Diakonos and P. Schmelcher, Phys. Lett. A **211**, 199 (1996).
[8] G. Györgyi and P. Szepfalusy, Z. Phys. B **55**, 179 (1984).
[9] G. Györgyi and P. Szepfalusy, J. Stat. Phys. **34**, 451 (1984).
[10] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).
[11] H. Mori, B. So, T. Ose, Prog. Theor. Phys. **66**, 1266 (1981).